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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS. (U)

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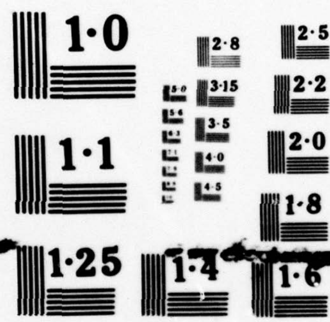

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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS

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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS

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May, 1978

ABSTRACT

A generalized critical point is characterized by the vanishing of certain linear relationships. In particular, the dynamics near such a point are completely non-linear. ~~In~~ this paper~~, we~~ analyzes fluctuations at such points of spatially homogeneous systems. ~~We discuss~~ thermodynamic critical points are discussed as a special case; but the main emphasis is on stochastic kinetic equations. ~~We show that,~~ fluctuations at a critical point cannot be characterized by a Gaussian density, but more sophisticated densities yield reasonable results. The ~~Our~~ theory is applied to the critical harmonic oscillator.

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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS

SECTION 1. INTRODUCTION

A generalized critical point can be characterized by the vanishing of certain linear relationships. Such a point may be thermodynamic or kinetic. For example, at the liquid-vapor transition temperature, it is well known that

$$\left.\frac{\partial P}{\partial V}\right|_{T_c} = 0 \quad . \tag{1.1}$$

In general, if $\Phi(x)$ is some generalized potential and $\Phi(0) = 0$ then the Onsager theory of non-equilibrium processes indicates that perturbations from $x = 0$ evolve according to

$$\dot{x} = -L \frac{\partial \Phi}{\partial x} \, . \tag{1.2}$$

In the vicinity of equilibrium, x_{eq}, since $\Phi'(x_{eq}) = 0$,

$$\Phi(x) = \Phi_{eq} + \frac{\Phi''(x_{eq})(x - x_{eq})^2}{2} + \ldots \tag{1.3}$$

Defining $1/\chi = \Phi''(x_{eq})$ to be the recriprocal of a "susceptibility" we find

$$\dot{x} = \frac{-L}{\chi}(x - x_{eq}) + O\left((x - x_{eq})^2\right) . \tag{1.4}$$

At a generalized critical point, $\Phi'' = 0$. Hence the dynamics in the vicinity of a critical point are highly nonlinear. Namely, one must use extra terms in the Taylor expansion (1.3), and obtain nonlinear dynamics at the critical point.

In this paper, we consider thermodynamic and kinetic critical points. As the term is usually used, "critical point" refers to a point in parameter space that is characterized by i) a slowing down of the dynamics and ii) long range spatial correlations. In this paper, we do not consider the long range order, but restrict the problem to spatially homogeneous systems, so that only critical slowing down will be evidenced. These equilibria (thermodynamic case) or steady states (kinetic case) are characterized by nonlinear dynamics. This point is given much emphasis in the paper. Due to the nonlinear dynamics the analysis of systems near critical points is quite difficult. Some analysis has been done, by Kubo et. al.(1) and Nitzan et. al.(2). In the first paper, two types of critical steady states were defined for one dimensional systems. We briefly review these. Let $x=0$ be the steady state. The steady state is of the marginal type(1) if perturbations from $x=0$ behave as

$$\dot{x} \sim \pm x^2. \tag{1.5}$$

The origin is a critical type steady state(1) if perturbations from x=0 behave as

$$\dot{x} \sim \pm x^3 . \tag{1.6}$$

Recently, the terminology of Kubo et. al. has been generalized to multidimensional systems(3). The paper of Nitzan et. al.(2) is contained as a special case of this paper and the accompanying one(4).

In this paper, we will analyze fluctuations at critical points. Often one reads that fluctuations "become unbounded" or "grow anomalously" at critical points. These statements are meant in the following sense: if one tries to describe fluctuations at a critical point by a Gaussian approximation, then the second moment $\langle x^2 \rangle$ is infinite. We will show that the Gaussian description of fluctuations implicitly assumes linear dynamics. Since critical point dynamics are nonlinear, one should not expect the Gaussian approximation to be valid. Hence, the anomaly is not in the physics, but in the improper use of mathematical approximations. We show that although the Gaussian approximation is not valid, more complicated densities are appropriate. The technique to demonstrate this will utilize formal asymptotic methods.

Our results are analogous to problems in optics (at a caustic) and wave mechanics (at a classical turning point). In those cases, the geometrical optics and WKB solutions break down, yielding infinite amplitudes. In reality, the intensity of light at a caustic is not infinite, but is large(5). At a caustic, geometrical optics must be replaced by Airy or Pearcey integrals(5,6). Similar analyses hold at the classical turning point (e.g.(7),(8)).

In §2, we give the uniform treatment of fluctuations at the thermodynamic critical point of a homogeneous system. As an example, we calculate volume fluctuations of van der Waals gas at the critical point. Our theory is equivalent to the classical theory of phase transitions(8). The main focus of this paper, however, is kinetic critical points, which have a much richer dynamical behavior. In §3, we introduce the stochastic kinetic equation and diffusion approximation. The theory given here is a variation of the mode-mode coupling theory(10). We discuss a possible resolution of the present controversy regarding a "proper" expansion of the Master Equation to obtain a Fokker-Planck equation(11). A small parameter arises in the derivation of the diffusion approximation; it characterizes the intensity of fluctuations. The fluctuations are characterized by a density that satisfies the forward or Fokker-Planck equation. In this paper, techniques for the construction of solutions of the forward equation are given when the underlying deterministic dynamics exhibit critical behavoir. In §4 we derive solutions of the one dimensional time invariant Fokker-Planck equation. We obtain an exact result, which is then analyzed by asymptotic methods. We obtain a Gaussian density at a non-critical steady state, an Airy density at a marginal type steady state, and a Pearcey density at a critical type steady state. In §5, these densities are used in a general ansatz ("ray method"(12)) to provide asymptotic solutions of the time dependent multidimensional Fokker-Planck equation. We construct densities in which susceptibilities (i.e. first derivatives) at the critical point are large, but finite. The same result applies to variances. In §6, we show how our results can be used to construct time-dependent correlation functions. In §7, we discuss an example of the critical harmonic oscillator(13) and show how the correlation function is constructed.

Antecedents to this work are found in Kubo et. al.(1), Kitahara(14), Keizer(15) and Nitzan et. al.(2). The present work generalizes the results of the above papers.

SECTION 2. UNIFORM THERMODYNAMIC THEORY

In this section, we derive the uniform theory for thermodynamic critical points of spatially homogeneous systems. The ideas which arise here are very similar to the more complex ones that arise in the kinetic case.

2.1. GENERAL THEORY

The thermodynamic theory proceeds from the Einstein fluctuation-entropy formula. We assume that the entropy of the system can be characterized by a parameter x. The equilibrium entropy is $S_0 = S(x_0)$, Let

$$v(x)dx = \Pr\left\{\text{system reaches a state in which } x \in (x, x+dx)\right\}. \tag{2.1}$$

Then(9)

$$v(x)dx = c \exp\left[\frac{S(x) - S_0}{k}\right]dx = c \exp\left[\frac{\Delta S}{k}\right] dx, \tag{2.2}$$

where c is a normalizing constant. As is usually done, we have eliminated the time variable from the discussion of thermodynamic problems(16). This elimination has certain conceptual drawbacks when one tries to describe the time evolution of a system. However, we shall follow standard notation here. We also follow the standard procedure of dividing the universe into the sub-system of interest, characterized by a variable y and certain intensive parameters $(\alpha_1, \ldots \alpha_n) = \alpha$, and an external reservoir which is assumed to remain in thermodynamic equilibrium(17, page 274). Then

$$v(y)dy = \tilde{c} \exp\left[\frac{-\Delta W(y)}{kT}\right] dy, \tag{2.3}$$

where $\Delta W(y)$ is the work done on the subsystem by an external source. Let $\Phi(y,\alpha)$ denote the potential of the system, so that (with $y=y_0$ denoting equilibrium):

$$\Delta W(y) = \Phi(y,\alpha) - \Phi(y_0,\alpha). \tag{2.4}$$

Then

$$v(y)dy = c \exp\left[\frac{-\Phi(y,\alpha)}{kT}\right] dy. \tag{2.5}$$

Usually (9,16,17) $\Phi(y,\alpha)$ is expanded in a Taylor series, keeping terms of second order

$$\Phi(y,\alpha) = \Phi(y_0,\alpha) + \Phi'(y_0,\alpha)(y-y_0)$$

$$+ \Phi'' \frac{(y_0,\alpha)}{2} (y-y_0)^2 + \ldots \tag{2.6}$$

At equilibrium $\Phi'(y_0,\alpha) = 0$ and $\Phi''(y_0,\alpha) > 0$. Thus we obtain

$$v(y)dy \sim \exp\left[\frac{-1}{2kT} \Phi''(y_0,\alpha)(y-y_0)^2\right] dy. \tag{2.7}$$

Equation (2.7) gives a locally Gaussian density with variance

$$\sigma^2 = \frac{kT}{\Phi''(y_0,\alpha)} \quad . \tag{2.8}$$

The Onsager-Machlup theory of irreversible processes(18) proceeds from this point.

Suppose, however, that there is a value $\alpha=\alpha_m$ such that

$$\Phi'(y_0,\alpha_m) = \Phi''(y_0,\alpha_m) = 0 \qquad \Phi'''(y_0,\alpha_m) \neq 0. \tag{2.9}$$

Such a point might correspond to a second order phase transition(2,9). A third term is needed in the Taylor expansion (2.6). For α near α_m, instead of (2.7), one obtains

$$v(y)dy \sim \exp\left[\frac{-1}{kT}\left\{\frac{\Phi''(y_0,\alpha)}{2}(y-y_0)^2 + \frac{\Phi'''(y_0,\alpha)}{6}(y-y_0)^3\right\}\right] dy. \tag{2.10}$$

A simple change of variables converts (2.10) to

$$v(y)dy \sim \exp\left[\frac{-1}{kT}\left\{z(y)^3 - \tilde{\alpha}z(y) + \beta\right\}\right] dz(y) \; , \tag{2.11}$$

where z(y) is a regular function of y and $\tilde{\alpha}(\alpha)$ is a regular function of α with the property that $\tilde{\alpha}(\alpha_m) = 0$. A density of the form (2.11) is called an Airy density.

At thermodynamic critical points, if $\Phi''(y_0,\alpha)$ vanishes, then usually $\Phi'''(y_0,\alpha)$ also will vanish. This can be shown by using free energy arguments(17). Such will not be the case for kinetic equations, however.

It is also possible that at a different value of $\alpha, \alpha=\alpha_c$.

$$\Phi''(y_0,\alpha_c) = \Phi'''(y_0,\alpha_c) = 0;\ \Phi^{(iv)}(y_0,\alpha_c) \neq 0. \tag{2.12}$$

Such a point corresponds to a first order phase transition(2). We take another term in the Taylor expansion (2.6) and obtain, for α near α_c:

$$v(y)dy \sim \exp\left[\frac{-1}{kT}\left\{\frac{\Phi^{iv}(y_0,\alpha)(y-y_0)^4}{24} + \frac{\Phi'''(y_o,\alpha)(y-y_o)^3}{6} + \frac{\Phi''(y_0,\alpha)(y-y_0)^2}{2}\right\}\right]dy. \tag{2.13}$$

Equation (2.13) can be put into the form

$$v(y)dy \sim \exp\left[\frac{-1}{kT}\left\{\frac{z(y)^4}{4} + \frac{\tilde{\alpha}_1(\alpha)z(y)^2}{2} + \tilde{\alpha}_2(\alpha)\right\}\right]dz(y). \tag{2.14}$$

In (2.14), $z(y)$ is a regular function of y; $\tilde{\alpha}_1, \tilde{\alpha}_2$ are regular functions of α and vanish at $\alpha=\alpha_c$. We call the density (2.14) a Pearcey density.

Equations (2.10,14) represent a formal extension of Onsager's theory to critical point phenomena. In light of (1.4), we are lead to totally nonlinear dynamics when $\alpha=\alpha_c$ or $\alpha=\alpha_m$. Our result is, of course, purely formal and is applicable to small deviations from equilibrium only, which is the best that one can expect of a thermodynamic theory.

2.2. VOLUME FLUCTUATIONS OF A VAN DER WAALS GAS

As an example of the above analysis, we consider the volume fluctuations of a gas at the liquid-vapor critical point, using a van der Waals model. Levich(17) shows that in this case

$$\Delta W = P_0 \Delta V + \Delta F, \tag{2.15}$$

where P_0 is the equilibrium pressure of the reservoir. Expanding ΔF gives

$$\Delta W = P_0 \Delta V + \left(\frac{\partial F}{\partial V}\right)_T \Delta V + \left(\frac{\partial^2 F}{\partial V^2}\right)_T \frac{(\Delta V)^2}{2} + \left(\frac{\partial^3 F}{\partial V^3}\right)_T \frac{(\Delta V)^3}{6} + \left(\frac{\partial^2 F}{\partial V^4}\right)_T \frac{(\Delta V)^4}{24} \quad , \tag{2.16}$$

where $\Delta V = V - V_0$ is the deviation from the equilibrium volume. Usually, only two terms are of (2.16) are used. If we set $-\left(\frac{\partial F}{\partial V}\right)_T = P = P_0$, we find

$$E\left((\Delta V)^2\right) = \frac{kT}{|(\partial P/\partial V)_T|} \ . \qquad (2.17)$$

The value of $E\,(\Delta V)^2$ becomes infinite at $T=T_c$. The cause of the divergence is purely mathematical: only two terms of the expansion were used. At the critical temperature(17)

$$\left(\frac{\partial P}{\partial V}\right)_{T_c} = \left(\frac{\partial^2 P}{\partial V^2}\right)_{T_c} = 0 \text{ but } \left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \neq 0. \qquad (2.18)$$

Then (2.16) becomes

$$\Delta W = -\left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \frac{(\Delta V)^4}{24} \ . \qquad (2.19)$$

Equation (2.19) is general. We now specialize to a van der Waals gas, for which

$$P = \frac{RT}{V-b} - \frac{a}{V^2} \ . \qquad (2.20)$$

The conditions (2.18) lead to the following values of the critical parameters:

$$T_c = \frac{8a}{27bR} \qquad V_c = 3b \qquad P_c = \frac{a}{27b^2} \ . \qquad (2.21)$$

Then

$$\left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \equiv -\gamma = \frac{.0123457a}{b^5} \; . \tag{2.22}$$

Then we obtain

$$v(\Delta V)\,d(\Delta V) = c \exp\left[\frac{-\gamma(\Delta V)^4}{24kT_c}\right] d(\Delta v), \tag{2.23}$$

where c is the normalization constant. Hence we obtain

$$E\left((\Delta V)^2\right) = \frac{\int_{-\infty}^{\infty} x^2 \exp\left[\frac{-\gamma x^4}{24kT_c}\right] dx}{\int_{-\infty}^{\infty} \exp\left[\frac{-\gamma x^4}{24kT_c}\right] dx}$$

$$= \left(\frac{kT_c}{\gamma}\right)^{1/2} \frac{\int_{-\infty}^{\infty} y^2 e^{-y^4/24}\, dy}{\int_{-\infty}^{\infty} e^{-y^4/24}\, dy} \; . \tag{2.24}$$

Thus, the van der Waals theory (i.e. the classical theory, see below) predicts $E\left((\Delta V)^2\right) \propto T_c^{1/2}$. In Table 1, we give the results of calculations of $E\left((\Delta V)^2\right)$ for a number of gases. The above theory yields values of $E\left((\Delta V)^2\right)$ which are large, but not infinite.

The thermodynamic theory presented in this section is a "classical" theory(19) and thus will not predict the "correct" critical exponents. This is a fault of the use of thermodynamic theory per se. It is not clear how the thermodynamic theory given above could be modified to yield the correct exponents.In a recent work Mou et. al.(34), using the master equation, also derived the classical result.

TABLE 1

VOLUME FLUCTUATIONS AT THE CRITICAL POINT

Gas	V_c(cc/mole)*	$E\left((\Delta V)^2\right)/V_c$
He	57.6	$4.416\text{x}10^2$
H_2	65.0	$4.336\text{x}10^2$
N_2	90.0	$4.832\text{x}10^2$
CO	90.0	$5.060\text{x}10^2$
O_2	74.4	$4.702\text{x}10^2$
C_2H_4	127.5	$5.168\text{x}10^2$
CO_2	95.7	$5.147\text{x}10^2$
NH_3	72.4	$6.780\text{x}10^2$
H_2O	45	$1.187\text{x}10^2$

*from (20, page 18)

SECTION 3. STOCHASTIC KINETIC EQUATIONS, DIFFUSION APPROXIMATION AND FOKKER-PLANCK EQUATION

The thermodynamic theory of §2 can not be used to treat highly nonequilibrium kinetic phenomena, which are of interest in many areas of chemistry, physics and biology. Let $\tilde{x}(t)$ denote the statistical variables. Often we can postulate an equation for the mean value of $\tilde{x}(t)$, $x(t)$:

$$\frac{dx^i}{dt} = b^i(x,\alpha) \qquad x^i(0) = x_0^i \qquad i = 1,\ldots,n$$

$$\alpha = \left\{\alpha_1, \ldots \alpha_m\right\}. \tag{3.1}$$

In order to treat fluctuations, we need to know the kinetic equation that $\tilde{x}(t)$ satisfies. Ideally, we would start with the Liouville equation and derive the kinetic equation. Such a derivation is possible for only the simplest system(33). Instead, we shall use a generalization of the Langevin method. We will add a zero-mean stochastic term to (3.1). The stochastic function $\tilde{Y}(\tau)$ is characterized by a microscopic time scale, τ, small compared to the macroscopic scale on which measurements are made. Hence

$$\Delta\tau = \eta^2 \Delta t \tag{3.2}$$

where η is a small parameter. We will not assume that $\tilde{Y}$ has a δ-correlation function and let

$$\gamma^{kl} = \int_0^{\infty} E\left(\tilde{Y}^k(s)\tilde{Y}^l(o)\right) ds.$$

We assume that $\tilde{x}(t) = \tilde{x}_\eta(t)$ satifies the stochastic kinetic equation

$$\frac{dx^i_\eta}{dt} = b^i(\tilde{x}_\eta) + \frac{\sqrt{\varepsilon}}{\eta} f^i_j(\tilde{x}_\eta)\tilde{Y}^j(t/\eta^2) \quad . \tag{3.3}$$

In equation (3.3), ε is a small parameter characterizing the size of the system and related to the intensity of the fluctuations (1,2,13,15). Hence $\varepsilon \to 0$ corresponds to the thermodynamic limit. The field $f^i_j(x)$ is a given deterministic field. Ideally, one would like to calculate f^i_j from basic principles. Since (3.3) is somewhat ad-hoc, a prescription is needed for the calculation of f^i_j. (One such prescription is the fluctuation-description theorem. Another is given by Keizer(22)). As $\eta \to 0$, $x_\eta(t) \to x(t)$, a diffusion process (22). We set

$$u(x) = E_x\left\{u_o\left(\tilde{x}(t)\right) \Big| \tilde{x}(0) = x\right\} . \tag{3.4}$$

Then u(x) satisfies

$$\frac{\partial u}{\partial t} = \frac{\varepsilon a^{ij}}{2} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + c^i \varepsilon \frac{\partial u}{\partial x^i} \equiv Lu \tag{3.5}$$

In (3.5), we use the convention that repeated indices are summed from 1 to n and

$$a^{ij} = f^i_k f^i_l (\gamma^{kl} + \gamma^{lk}) \tag{3.6}$$

$$c^i = \gamma^{kl} f^j_k \frac{\partial f^i_l}{\partial x^j} = \frac{1}{4} \frac{\partial}{\partial x^j} a^{ij}. \tag{3.7}$$

On the other hand, if $\tilde{Y}$ has a δ-correlation function (white noise), then we obtain

$$\frac{\partial u}{\partial t} = \frac{\varepsilon a^{ij}}{2} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} . \tag{3.8}$$

Numerical work(3) indicates that if the boundaries are non-singular, then equation (3.5) and (3.8) yield equivalent solutions for $\varepsilon<.1$.

We also note that equation (3.3) is a stochastic equation with correlations and hence is a more reasonable representation then a white noise equation. Furthermore, equation (3.5) [or (3.8)] is derived rigorously--no expansion procedure is needed (compare(11)). Equation (3.8) is the backward equation. Usually, in the physical literature the forward or Fokker-Planck equation is used. This equation can not be derived rigorously. No expansion procedure will rigorously give the Fokker-Planck equation(1,11). Instead, we will obtain the Fokker-Planck equation by using the theory of partial differential equations. In (24, 25) it is shown now this can be achieved. Let

$$v(x,t)dx = \Pr\left\{x \leq \tilde{x}(t) \leq x+dx\right\} . \tag{3.9}$$

Then $v(x,t)$ satifies (at least weakly) the adjoint equation

$$v_t = L^* v = \frac{\varepsilon}{2}(a^{ij}v)_{ij} - (b^i v)_i - \varepsilon(c^i v)_i . \tag{3.10}$$

In the derivation of (3.10), there is a question of boundary terms for u, v as $|x|\to\infty$(25). For the problems considered here these questions are relatively unimportant. In (3.10) subscripts indicate differentiation.

SECTION 4. CANONICAL DENSITIES

In this section, we consider the time independent, one dimensional Fokker-Planck equation

$$\varepsilon\frac{(av)_{xx}}{2} - (bv)_x = 0 \tag{4.1}$$

subject to

$$\int_{-\infty}^{\infty} v(s)\,ds = 1 \qquad \lim_{|s|\to\infty} v(s) = 0. \tag{4.2}$$

The solution of (4.1,2) gives the steady state (but not necessarily equilibrium) density for a process satisfying (3.3). Our results are valid for $\varepsilon\to 0$ ("thermodynamic limit") and generalize the thermodynamic results of §2. In later sections, we generalize the solutions obtained here to solve time dependent, multidimensional problems.

When (4.1) is integrated twice and (4.2) is applied, we find

$$v(x) = k\left[\exp\left\{\int^{x} \frac{2b}{\varepsilon a}\,ds\right\}\right], \tag{4.3}$$

where k is the normalization constant

$$k = \int_{-\infty}^{\infty} \exp\left\{\int^{x} \frac{2b}{\varepsilon a}\,ds\right\} dx. \tag{4.4}$$

The main contribution to (4.3) comes from the maximum of the function

$$\Phi(x) = \int^{x} \frac{2b}{a}\, ds. \tag{4.5}$$

We now assume that there is a steady state (i.e. $b(x)=0$ has a solution), x_0. The steady state is classified according to its dynamic behavior.

The <u>normal type</u> steady state x_0 is characterized by

$$b(x_0) = 0, \quad b'(x_0) \neq 0. \tag{4.6}$$

We are interested in stable steady states, so that we assume $b'(x_0) < 0$. Thus, perturbations from x_0 decay exponentially. When $\Phi(x)$ is expanded about x_0, we obtain:

$$v(x) \sim k \left[\exp\left\{ \frac{-|b'(x_0)|(x-x_0)^2}{\varepsilon a(x_0)} \right\} \right] + o\left[\exp\left\{ \frac{-|b'(x_0)|(x-x_0)^2}{\varepsilon a(x_0)} \right\} \right]. \tag{4.7}$$

Thus, we obtain a locally Gaussian density, for small ε. This result has also been derived by Kubo et. al.(1) and Keizer(23) by different arguments. It is the standard result in the theory of nonequilibrium thermodynamics (18).

In the marginal case, b depends on one parameter α such that when $\alpha = \alpha_c$ the marginal type steady state satisfies

$$b(x_0,\alpha_c) = 0 \qquad b'(x_0,\alpha_c) = 0 \qquad b''(x_0,\alpha_c) \neq 0. \tag{4.8}$$

The canonical dynamics corresponding to the marginal case are (4)

$$\dot{x} = x^2 - \alpha. \tag{4.9}$$

The flow of such dynamics is sketched in Fig. 1. We need to replace the conditions (4.2) by:

$$\lim_{s\to-\infty} v(s) = 0, \quad \int_{-\infty}^{x_F} v(s)ds = 1 \tag{4.10}$$

where $x_F < \infty$ is an end value for x (see also 3).

Since $b'(x_0,\alpha_c) = 0$, the expansion used to obtain (4.7) breaks down. Hence the Gaussian density breaks down. In particular, from (4.7) we have

$$E(x^2) \propto 1/|b'(x_0,\alpha|. \tag{4.11}$$

Thus, when $\alpha = \alpha_c$, the Gaussian density yields an infinite variance. This is, of course, a purely mathematical divergence and has nothing to do with the physical problem. We take an extra term in the expansion of $\Phi(x,\alpha)$ to obtain

$$v(x) \sim k \exp\left[\frac{b'(x_0,\alpha)(x-x_0)^2}{\varepsilon a(x_0)} + \frac{b''(x_0,\alpha)(x-x_0)^3}{3\varepsilon a}\right]. \qquad (4.12)$$

A change of variables converts (4.12) to the Airy density (2.11).

The critical type steady state is characterized by two parameters, α,β such that when $\alpha = \alpha_c$ and $\beta = \beta_c$

$$b(x_0,\alpha_c,\beta_c) = b'(x_0,\alpha_c\beta_c) = b''(x_0,\alpha_c,\beta_c) = 0$$

$$b'''(x_0,\alpha_c,\beta_c) \neq 0. \qquad (4.13)$$

The canonical dynamics of a critical type dynamical system are (4)

$$\dot{x} = \pm x^3 + \alpha x + \beta. \qquad (4.14)$$

In this case $\alpha_c = \beta_c = 0$. The flow of (4.14) is skeched in Fig. 1.

It is clear that the Airy and Gaussian densities both break down when $\alpha = \alpha_c$ and $\beta = \beta_c$. In this case, we take one more term in the Taylor expansion of $\Phi(x)$ and obtain the Pearcy density (2.14).

The results of this section can be obtained by direct use of Levinson's theorem(26). It is clear that the Gaussion approximation will be valid whenever $|b'(x_0,\alpha)/b''(x_0,\alpha)| >> 1$.

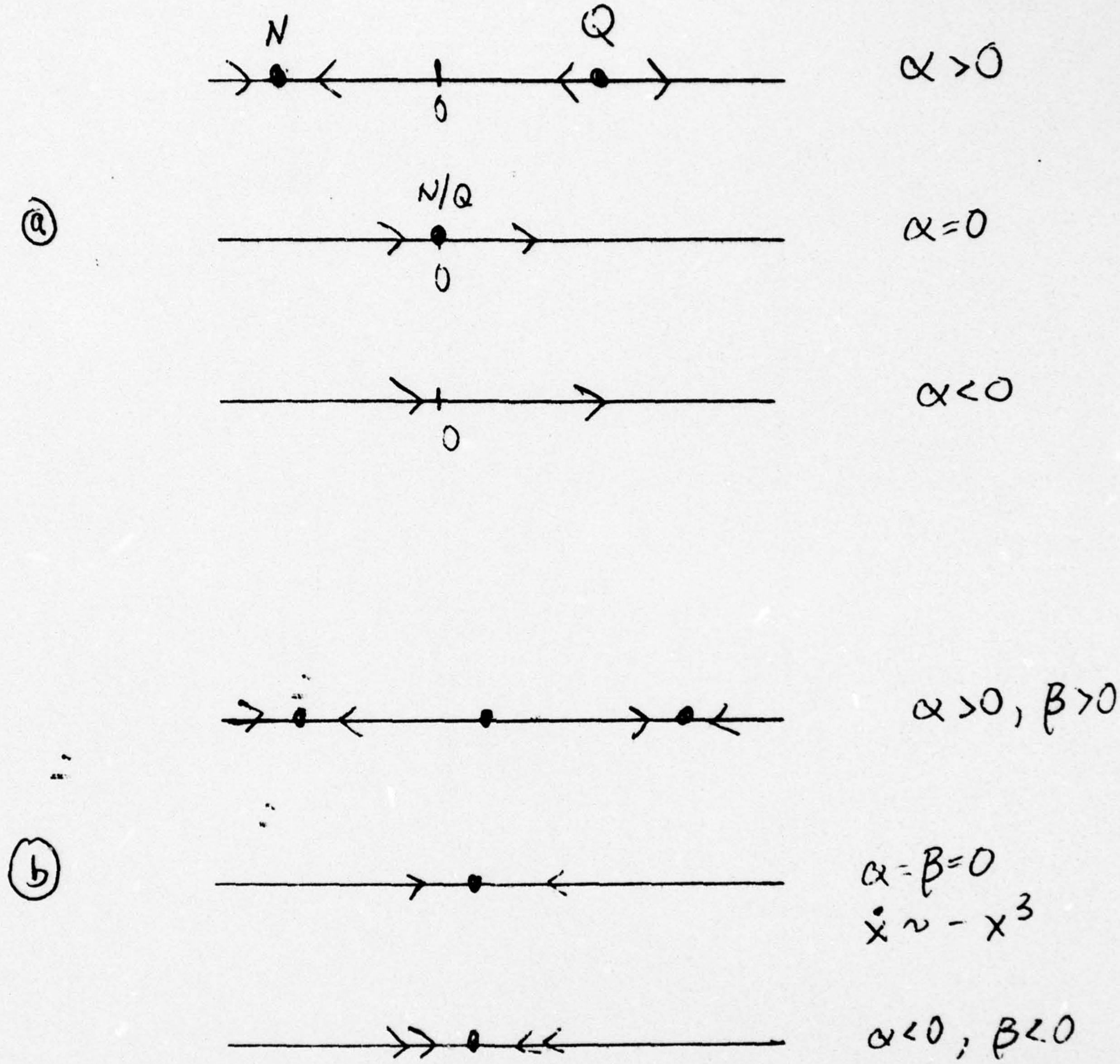

FIG. 1: A) THE MARGINAL TYPE DYNAMICAL SYSTEM HAS TWO STEADY STATES WHEN A PARAMETER $\alpha > 0$, ONE DEGENERATE STEADY STATE WHEN $\alpha = 0$ AND NO STEADY STATES WHEN $\alpha < 0$.
B) THE CRITICAL TYPE DYNAMICAL SYSTEM HAS THREE STEADY STATES WHEN $\alpha > 0$, $\beta > 0$, ONE DEGENERATE STEADY STATE WHEN $\alpha = \beta = 0$ AND ONE STEADY STATE WHEN $\alpha > 0$, $\beta > 0$.

SECTION 5.

TIME DEPENDENT, MULTIDIMENSIONAL FOKKER-PLANCK EQUATION

In this section, we construct regular solutions of the time dependent Fokker-Planck equation (3.5). Since Ludwig(5) has given the construction for the normal case, we only consider marginal and critical type steady states. Our goal is to construct densities that have finite second moments. Exact definitions of marginal and critical type steady states in multidimensional systems are given in Appendix A.

5.1. MARGINAL TYPE STEADY STATE

We seek a solution of (3.5) of the form

$$v(x,t) = \exp\left[\frac{-1}{\varepsilon}\left(\frac{\Psi(x,t)^3}{3} - \tilde{\alpha}\,\Psi(x,t)\right)\right]\sum_{n=0} \varepsilon^n z^n(x,t). \tag{5.1}$$

The form of (5.1) is a "ray ansatz"(27). In it, $\Psi(x,t)$, α, and the functions $z^0(x,t), z^1(x,t)\ldots$ are to be determined. In practice we are often interested in just the first term of (5.1). After derivatives are evaluated, terms are collected according to powers of ε. We obtain

$$0 = \exp\left[-\frac{1}{\varepsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right]/\varepsilon \left\{\Psi_t + b^i\Psi_i\right.$$

$$\left. + (\tilde{\alpha} - \Psi^2)\,\frac{a^{ij}}{2}\,\Psi_i\Psi_j\right\} z^0(\tilde{\alpha} - \Psi^2) + \exp\left[-\frac{1}{\varepsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right] \left\{b^i_{,j} z^0\right.$$

$$+ b^i z^0_i + \frac{a^{ii}_{,ij}}{2} z^0 + \frac{a^{ij}_{,j}}{2}(\tilde{\alpha} - \Psi^2)\Psi_i z^0 \tag{5.2}$$

$$z_t + a^{ij}\left(-2\Psi\Psi_i\Psi_j z^0 + (\tilde{\alpha} - \Psi^2)\ \Psi_{ij} z^0\right.$$

$$\left.\left. + 2(\tilde{\alpha} - \Psi^2)\Psi_j z^0_i\right) - c^i(\tilde{\alpha} - \Psi^2)\Psi_i z^0\right)\right\}$$

$$+ 0\left(\varepsilon \exp\left[-\frac{1}{\varepsilon}\left(\frac{\Psi^3}{3} - \tilde{\alpha}\Psi\right)\right]\right) .$$

The leading coefficient of ε vanishes if

$$\Psi_t + b^i\Psi_i + \frac{(\tilde{\alpha} - \Psi^2)}{2}\, a^{ij}\Psi_i\Psi_j = 0 \tag{5.3}$$

Equation (5.3) is a generalized eikonal equation(3). In the rest of this paper, we shall assume that the initial data for v are concentrated at a point

$$v(x,0) = \delta(x - x_0) \tag{5.4}$$

At the deterministic steady states, N,Q (see Fig. 2) we expect that $d\Psi/dt = \Psi_t + b^i\Psi_i = 0$. We set $\Psi^2 = \tilde{\alpha}$ at those points. The

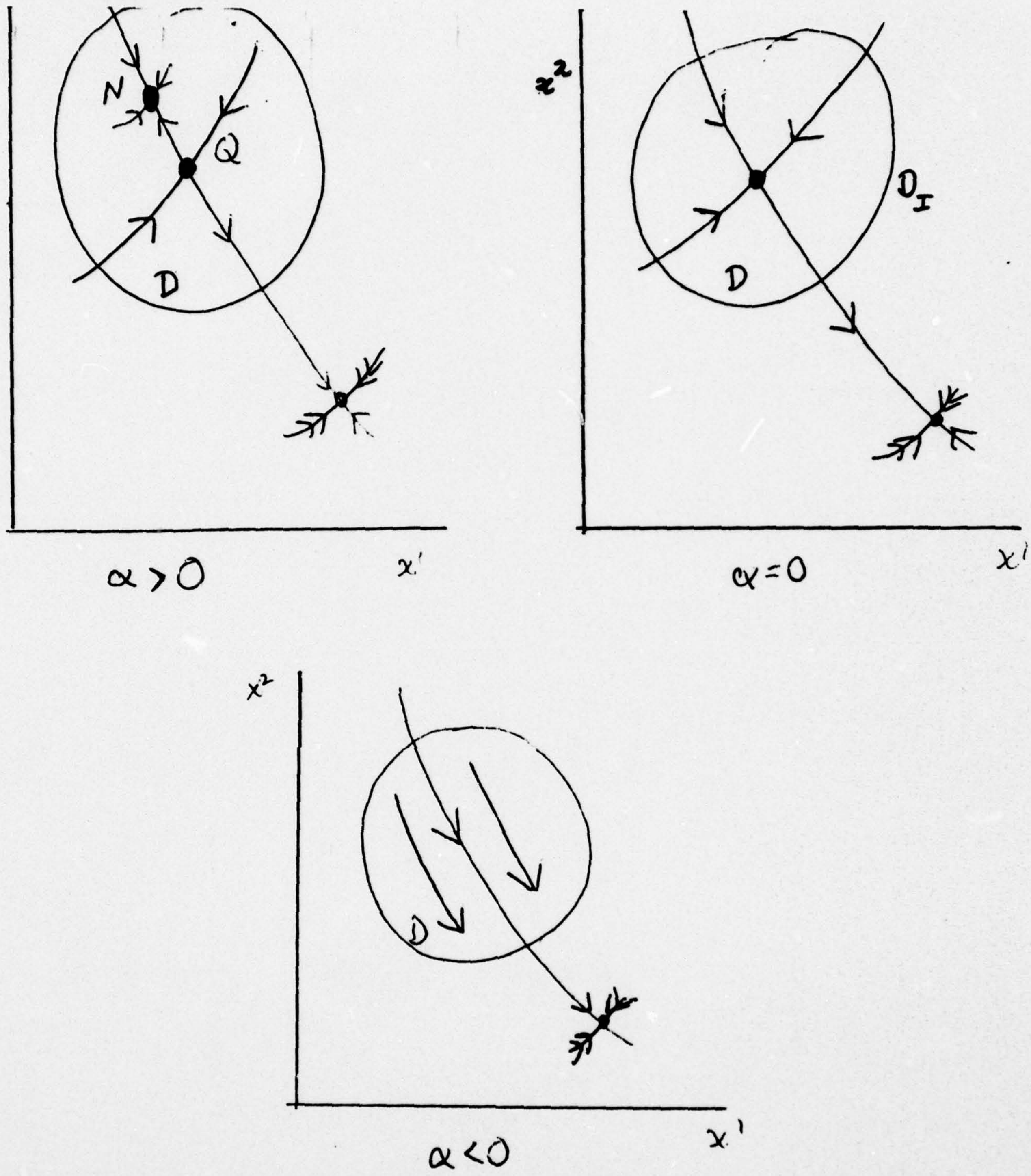

FIG. 2: A TWO DIMENSIONAL MARGINAL DYNAMICAL SYSTEM IN A DOMAIN D IN R^2. WHEN $\alpha < 0$ THE DETERMINISTIC FLOW IS ALWAYS ACROSS D, A PROBLEM FIRST STUDIED BY LEVINSON(28)

node N should correspond to a local maximum for $v(x,t)$. Hence we set $\Psi(N) = \sqrt{\tilde{\alpha}}$. Similar reasoning leads to $\Psi(Q) = -\sqrt{\tilde{\alpha}}$.

The value of α is still undertermined. It can be obtained by the following iterative procedure(3). (If higher order terms are to be considered, then it is necessary to expand $\tilde{\alpha} = \sum_{k=0} \varepsilon^k \tilde{\alpha}_k$. In that case, all of the parameters are determined in a manner analogous to the determination of $\tilde{\alpha}$). We start at the node N, where $\Psi = \sqrt{\tilde{\alpha}^0}$, the first estimate for $\tilde{\alpha}$. Equation (5.3) can be solved by the method of characteristics(12).The characteristic equations are:

$$\frac{dt}{ds} = 1 \tag{5.5a}$$

$$\frac{dx^i}{ds} = b^i + (\tilde{\alpha} - \Psi^2)p_j a^{ij} \tag{5.5b}$$

$$\frac{d\Psi}{ds} = p_k \frac{dx^k}{ds} = \frac{1}{2}(\tilde{\alpha} - \Psi^2)a^{ij}p_i p_j \tag{5.5c}$$

$$\frac{dp_k}{ds} = -\left(2p_k(-\Psi a^{ij}p_i p_j) + b^i_{,k}p_i + \frac{(\alpha - \Psi^2)}{2} a^{ij}_{,k}p_i p_j\right). \tag{5.5d}$$

Initial data is given on an ellipsoid surrounding N. As $x \to Q$, the value of α should approach $-\sqrt{\tilde{\alpha}^0}$. If it does not, then a new estimate $\tilde{\alpha}^{(1)}$ is needed. Elsewhere, we have shown that iterates of $\tilde{\alpha}$ can be determined by using the method of false position and that α is a regular function of the deterministic parameter α(3).

When N and Q coalesce (Fig. 2b), $\tilde{\alpha} = 0$. After the annihilation of N,Q (Fig. 2c), $\tilde{\alpha} < 0$. The stochastic problem for a dynamical system similar to Fig. 2c is an old one, solved by Levinson(28) and Ventcel and Friedlin(29). Consequently, we restrict ourselves to the dynamic cases represented by Figs. 2a,b.

From (5.5b), we see that if $\Psi^2 = \alpha$ on a trajectory, then $dx^i/ds = b^i$, so that the trajectory is a deterministic trajectory. In this way, we will be able to estimate deviations from a given deterministic trajectory. When (5.3) is differentiated with respect to x^k and evaluated on a trajectory we find:

$$\frac{d\Psi_k}{dt} + b^i_{,k}\Psi_k \mp \sqrt{\tilde{\alpha}}\,a^{ij}\Psi_i\Psi_j\Psi_k = 0 \qquad k = 1, 2, \ldots n \tag{5.6}$$

In (5.6) the (-) sign corresponds to trajectories that enter N, the (+) sign to trajectories that enter Q. At either of the steady states, we obtain

$$b^i_{,k}\Psi_k \mp \sqrt{\tilde{\alpha}}\; a^{ij}\Psi_i\Psi_j\Psi_k = 0 \qquad k = 1, \ldots n \tag{5.7}$$

Equation (5.7) can be solved to yield values of Ψ_k at N or Q.

When N and Q coalesce, so that $\alpha = 0$ and conditions (A) hold, it is possible to show that the Ψ_k can be calculated. For example, consider the case of only one spatial dimension. Then (5.7) becomes:

$$b,_x - \Psi(N)a\Psi_x^2 = 0 \tag{5.8}$$

or

$$\Psi_x^2(N) = \frac{b,_x}{\Psi(N)a} . \tag{5.9}$$

In obtaining (5.8,9), we have replaced $\sqrt{\tilde{\alpha}}$ by $\Psi(N)$. When N,Q coalesce, $b_x \to 0$ and $\Psi(N) \to 0$. One application of l'Hospital's rule gives

$$\Psi_x^3(N) = \frac{b,_{xx}(N)}{a(N)} . \tag{5.10}$$

A similar, but more complicated, calculation holds in the multi-dimensional cases(3).

Thus far, we have given our construction without any boundary conditions. In order to determine z^0, we need to specify the boundary conditions. As time progresses, the process will tend to concentrate (if it is still in D) near D_I. (Fig. 2).

The $0\left(\exp\left(\frac{1}{\varepsilon}\left(\frac{\Psi^3}{3} - \alpha\Psi\right)\right)\right)$ term in (5.2) yields a "transport" equation for z^0(25, 27). It takes the form

$$\frac{dz^0}{ds} + f(s)z^0 = 0, \tag{5.11}$$

i.e.

$$z^0(s) = z^0(0)\exp\left[-\int^s f(s')ds\right] \tag{5.11a}$$

When the initial data is concentrated at a point, Ludwig has shown that the appropriate initial data for z is $z^0(0)$ = constant.

5.2. CRITICAL TYPE STEADY STATE

For the critical type steady state, instead of (5.1), we seek a solution of (3.5) of the form

$$v(x,t) = \exp\left[-\frac{1}{\varepsilon}\left(\frac{1}{4}\Psi^4 - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right)\right]\sum_{n=0}\varepsilon^n z^n(x,t). \tag{5.12}$$

In this case, it is possible to impose the conditions on v(x,t) that

$$v\to 0 \text{ as } |x|\to\infty, \quad \int_{-\infty}^{\infty} v(x,t)dx = 1 \tag{5.13}$$

and take all of R^n as the domain of interest. Instead of (5.3), we obtain

$$\Psi_t + b^i\Psi_i + \frac{a^{ij}}{2}(\Psi^3 - \tilde{\alpha}\Psi - \beta)\Psi_i\Psi_j = 0. \tag{5.14}$$

The value of Ψ at the deterministic nodes N_1, N_2 and saddle Q (Fig. 3) is determined in a manner analogous to the one used in §5.1. The values of the parameters are also determined in a similar fashion. It is possible to show that all constructions remain regular as the steady states coalesce(3). The function $z^0(x,t)$ can also be determined in manner analogous to the previous case.

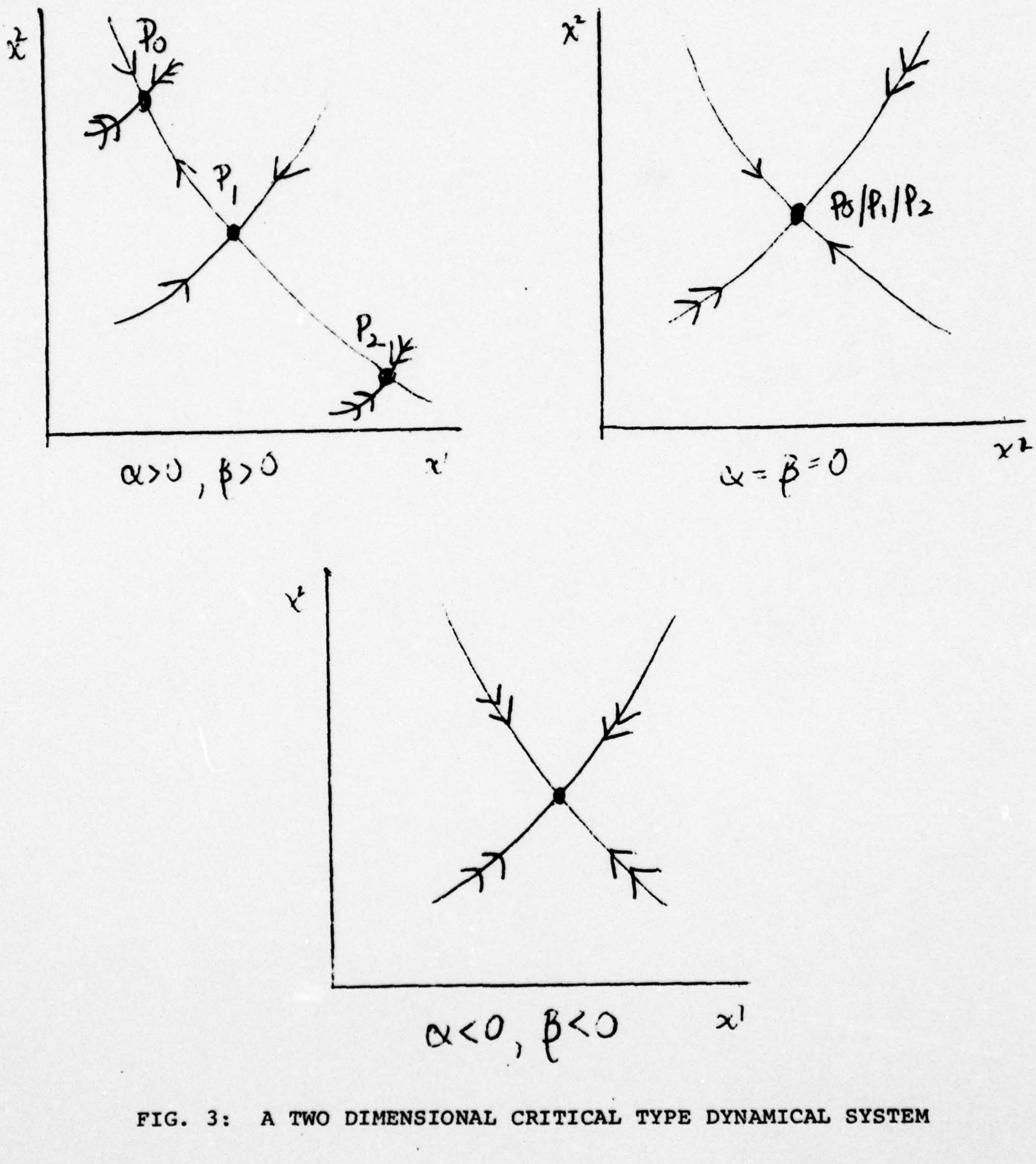

FIG. 3: A TWO DIMENSIONAL CRITICAL TYPE DYNAMICAL SYSTEM

SECTION 6. CORRELATION FUNCTIONS FOR CRITICAL TYPE SYSTEMS

In many physical problems, the object of interest is the correlation function

$$R(\tau) = E\left\{\tilde{x}(t)\ \tilde{x}(t+\tau)\right\} \tag{6.1}$$

$$= \int\int xx^1 \Pr\left\{\tilde{x}(t)\in(x,x+dx),\ \tilde{x}(t+\tau)\in(x^1,x^1+dx^1)\right\} dx dx^1.$$

Since the process in our problem is assumed to be stationary, $R(\tau) = E\left\{\tilde{x}(0)\tilde{x}(\tau)\right\}$. Now we consider the conditional correlation function:

$$R_{x_0}(\tau) = E\left\{\tilde{x}(0)\tilde{x}(\tau)\,|\,\tilde{x}(0) = x_0\right\}. \tag{6.2}$$

If $v_0(x_0)dx_0$ the initial density for x_0, then we clearly have

$$R(\tau) = \int R_{x_0}(\tau)v_0(x_0)dx_0. \tag{6.3}$$

However,

$$R_{x_0}(\tau) = \int x^1 \Pr\left\{\tilde{x}(\tau)\in(x^1,x^1+dx^1)\,|\,\tilde{x}(0) = x_0\right\} dx^1$$

$$= \int x^1\ v_{x_0}(x^1,\tau)dx^1, \tag{6.4}$$

where $v_{x_0}(x^1,\tau)dx^1$ was calculated in the previous section. Thus

$$R_{x_0}(\tau) = \int x^1 \exp\left[-\frac{1}{\varepsilon}\left(\frac{\Psi(x^1,\tau)^4}{4}\right) - \frac{\tilde{\alpha}\Psi^2}{2} - \tilde{\beta}\Psi\right] z(x,\tau)\Big|_{x(0) = x_0} dx \tag{6.5}$$

Namely, we start the ray calculation at $x = x_0$ and integrate the ray equations

$$\frac{dt}{ds} = 1 \qquad \frac{dx^i}{ds} = b^i + \Psi^3 a^{ij} p_j . \tag{6.6}$$

$$\frac{d\Psi}{ds} = p_k \frac{dx^k}{ds} \qquad \frac{dp_k}{ds} = -(3a^{ij} p_i p_j p_k + b^i_{,k} p_i$$

$$+ \Psi^3 \frac{a^{ij}_{,k}}{2} p_i p_j), \tag{6.7}$$

until $s = \tau$. Thus (6.5) can be evaluated. The full correlation function, obtained from (6.4) is

$$R(\tau) = \iint x^1 \exp\left[-\frac{1}{\varepsilon}\left(\frac{\Psi(x^1,\tau)^4}{4} - \frac{\tilde{\alpha}\Psi^2}{2}\right.\right.$$

$$\left.\left. - \tilde{\beta}\Psi\right] z^0(x,\tau)\Big|_{x(0) = x_0} v_0(x_0)\, dx^1 dx_0 . \tag{6.8}$$

In the next section, we give an example of such a calculation.

SECTION 7. CRITICAL HARMONIC OSCILLATOR

In this section, we consider a modified Duffing oscillator(30) in contact with a heat bath(13,31). We shall use a stochastic equation of the standard "mode-mode" form, but initially will indicate how a more general analysis would proceed.

The Hamiltonian of the system is

$$H = \frac{k(\eta)x^2}{2} + \frac{\alpha x^4}{4} + \frac{p^2}{2m} + \Phi_{int}(\underset{\sim}{r}, x) + \frac{p_i^2}{2m_i} \quad . \tag{7.1}$$

In (7.1), $\Phi_{int}(\underset{\sim}{r}, x)$ is the interaction potential of the oscillator with coordinates (x,p) and heat bath with coordinates (r^i, p_i). The last term represents the kinetic energy of the heat bath. The motion at the full system (oscillator + heat bath) is generated by

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{7.2}$$

$$\dot{r}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial r^i} \qquad i = 1, \ldots N$$

The motion of the entire system occurs on a manifold in the phase space given by $H = \overline{E}$, where E is the initial (i.e. constant) energy of the system. This manifold, M, will be bounded and compact. We expect that the full system is ergodic(32). We are interested in a submanifold of M, M^1, which is the manifold of (x,p) coordinates. A possible projection operator from M to M^1 is

$$\mathcal{P}\left\{x,p,r_j^i \ldots r^n, p_1, \ldots p_n\right\} = \left\{x,p,0, \ldots , 0\right\}. \tag{7.3}$$

Namely, we "project" from $M \to M^1$. On M^1, we assume that the following measure exists

$$\bar{P}(t,x,p,A) = \Pr\left\{(\tilde{x}(t),\tilde{p}(t))\in A \,|\, \tilde{x}(0)\in(x,x + dx),\ \tilde{p}(0)\in(p,p + dp)\right\}. \tag{7.4}$$

We have introduced $\tilde{x}(t)$, $\tilde{p}(t)$ as random variables. This is a result of the elimination of $(\underset{\sim}{r}, \underset{\sim}{p})$ from consideration. By averaging, we are treating the latter variables as random; thus x,p become random variables

Next, we assume that if $\tilde{x}(t) = x$, $\tilde{p}(t) = p$, then

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\int (\tilde{x} - x)\ \bar{P}\ (t + \Delta t,x,p,dx,dp) = p/m \tag{7.5}$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\int (\tilde{p} - p)\ \bar{P}\ (t + \Delta t,x,p,dx,dp) = -k(\eta)x - \alpha x^3 - \gamma(x)p \tag{7.6}$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\int (\tilde{p} - p)^2\ \bar{P}\ (t + \Delta t,x,p,dx,dp) = \varepsilon a, \tag{7.7}$$

and that all other moments are zero. These assumptions have yet to be verified for any but the simplest system(33). These assumptions lead to the Langevin equations

$$\dot{x} = p/m \tag{7.8}$$

$$\dot{p} = -k(\eta)x - \alpha x^3 - \gamma x^3 + \sqrt{\varepsilon a}\ \frac{d\tilde{y}}{dt}\ , \tag{7.9}$$

where $\tilde{y}(t)$ is an approximation to Gaussian white noise (e.g.(22), page 651).

The Fokker-Planck equation for the density $f(t,x,p)$ is

$$\varepsilon a\rho f_{pp} - \frac{p}{m} f_x - \Big((-k(\eta)x - \alpha x^3 - \gamma p)f\Big)_p = f_t\ , \tag{7.10}$$

where (see §3)

$$\rho = \int_0^\infty E\Big(\tilde{y}(s)\tilde{y}(0)\Big)\,ds. \tag{7.11}$$

The equilibrium density is ($\beta^{-1} = kT$)

$$f_{eq} = \exp\left[-\beta\left(\frac{p^2}{2m} + k(\eta)x + \frac{\alpha x^4}{4}\right)\right]. \tag{7.12}$$

We require that f_{eq} be a solution of (7.10) and obtain (the "fluctuation-dissipation" result):

$$\varepsilon\alpha\rho = \frac{2\gamma}{\beta} = 2kT\gamma. \tag{7.12}$$

The steady states of the averaged equations (7,8,9) are

$$p = 0 \qquad x = 0, \pm i\sqrt{|k(\eta)|_{\alpha}}, \tag{7.14}$$

where we have made the assumption that $k \geq 0$, $\alpha > 0$. We assume that when $\eta = \eta_c$, $k(\eta_c) = 0$. Then (0,0) is a critical type steady state. At $\eta = \eta_c$, we have a critical harmonic oscillator.

We now nondimensionalize (7.9). We let $\varepsilon = \frac{kT}{E_0} \ll 1$ be a small parameter, where E_0 is some reference energy. Introducing dimensionless variables by

$$v = \sqrt{\frac{E_0}{m}}\, v' \qquad x = \sqrt{\frac{E_0 m}{\gamma_o^2}}\, x' \qquad t = \frac{mt'}{\gamma_0} \quad , \tag{7.15}$$

$$k = \frac{\gamma}{\sqrt{E_0 m}}\, k' \qquad \alpha = \frac{\gamma^4}{(E_0 m)^{3/2}} \qquad \gamma(x) = \gamma'(x')\gamma_0$$

equation (7.9) becomes (with $v \equiv p/m$)

$$f_{t'} = \varepsilon\gamma' f_{v'v'} - (v'f)_{x'}$$

$$- \left((-k'x' - \alpha'(x')^3 - \gamma' v')f\right)_{v'} \quad . \tag{7.16}$$

In the sequel, we drop the primes in (7.16). Since $k' \propto k$, when $\eta = \eta_c$, $k'(\eta_c) = 0$.

We now seek a solution of (7.16) of the form

$$f(t,x,v) = \exp\left[-\frac{1}{\varepsilon}\left(\frac{1}{4}\Psi^4 - \frac{\alpha\Psi^2}{2} - \beta\Psi\right)\right]\sum \varepsilon^n z^n(x,t) \qquad (7.17)$$

where Ψ, α, β and $\bar{z}^n$ are to be determined. Following the procedure in §5, we obtain

$$\Psi_t + v\Psi_x - (k(\eta)x + \alpha x^3 + \gamma v)\,\Psi_v$$

$$+ \gamma(\Psi^3 - \alpha\Psi - \beta)\Psi_v^2 = 0. \qquad (7.18)$$

Let us now specialize to $\eta = \eta_c$, $k = 0$; i.e. the critical harmonic oscillator. Then $\alpha = \beta = 0$ in (7.17) and (7.18). The ray equations become

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\alpha x^3 - \gamma v + 2\gamma\Psi_v\Psi^3$$

$$\frac{d\Psi}{dt} = v\Psi_x + \frac{dv}{dt}\Psi_v + \Psi_t$$

$$\frac{d\Psi_x}{dt} = -6\Psi_x\Psi_v^2\Psi^2 - 3\breve{\alpha}x^2\Psi_v \qquad (7.19)$$

$$\frac{d\Psi_v}{dt} = -6\Psi_v^3\Psi^2 - \Psi_x + \gamma\Psi_v$$

By integrating the ray equations from an initial point $\tilde{x}(t_0) = x_0$, $v(t_0) = v_0$, we obtain the conditional density $f(x,t,v,x_0,v_0)$. Then following the procedure in §6, we can obtain the correlation function. Our solution thus allows the calculation of correlation functions at critical points. Once the correlation function is known, we obtain the spectrum of the oscillator by Fourier transform.

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APPENDIX A

MARGINAL AND CRITICAL TYPE DYNAMICAL SYSTEMS

In this appendix, we give exact conditions for marginal and critical type dynamical systems. Our work generalizes the scheme of Kubo et. al.(1973).

MARGINAL TYPE DYNAMICAL SYSTEMS

The deterministic evolution of the macrovariables is governed by

$$\dot{x} = b(x,\eta) \tag{A.1}$$

where $\eta\ R^1$ is a parameter. Equation (A.1) may have three steady states, $Q_0(\eta)$, $Q_1(\eta)$ and P_2. Let B_k be the matrix $(b^i{,}_j)$ evaluated at Q_0, Q_1 or P_2 $(k = 0,1,2)$. We assume that:

- For all values of η, B_2 has two real negative eigenvalues. Although P_2 may depend upon η, P_2 is always bounded away from the other steady states.
- As $\eta \downarrow 0$, the distance between $Q_0(\eta)$ and $Q_1(\eta)$ decreases. When $\eta = 0$, Q_0 and Q_1 coalesce and annihilate each other (i.e. when $\eta < 0$, (A.1) has one real and two complex steady states).
- When $\eta > 0$, B_0 has two real negative eigenvalues and B_1 has one real positive and one real negative eigenvalue. When $\eta = 0$, $b_0 = B_1$ has one zero and one real negative eigenvalue. The

eigenvector corresponding to the negative eigenvalue has positive slope. The double point $Q_0(0)/Q_1(0)$ is called a saddle node (3).

A deterministic system satisfying the above assumptions will be structurally similar to the system sketched in Fig. 2.

The above conditions can be reformulated by a change of coordinates. Define the y^1 axis in the direction of the eigenvector of the non-negative eigenvalue of B_1. The y^2 axis is in the direction of the eigenvector of the negative eigenvalue of B_1, with the origin at Q_1. Then

$$\dot{y} = \tilde{b}(y,\eta) \tag{A.2}$$

is the deterministic system in the new coordinates. The system is of the marginal type if:

$$\begin{aligned}
&1)\quad \det(\tilde{b}^i{}_{,j}(Q_1,0)) = 0 \\
&2)\quad \tilde{b}^1{}_{,1}(Q_1,0) = \tilde{b}^2{}_{,1}(Q_1,0) = 0 \\
&3)\quad \tilde{b}^2{}_{,2}(Q_1,0) \neq 0 \\
&4)\quad \tilde{b}^1{}_{,11}(Q_1,0) - \tilde{b}^2{}_{,11}(Q_1,0) \neq 0.
\end{aligned} \tag{A.3}$$

The conditions (A.3) have the following interpretation. Condition 1) indicates that the original system has a zero eigenvalue. Condition 2) indicates that when $\eta = 0$ the linear dynamics in the

y^1 direction vanish, condition 4) indicates that these dynamics are quadratic. Condition 3) indicates that the second eigenvalue is non-zero.

CRITICAL TYPE DYNAMICAL SYSTEMS

The macrovariables evolve according to a deterministic kinetic equation

$$\dot{x} = b(x,\eta,\delta) \tag{A.4}$$

where η, δ are one dimensional parameters. The entire bifurcation set of equation (A.4) is still unknown (3). The physical systems of interest here motivate the following assumptions:

- For some values of η, δ, (A.4) has three steady states $P_0(\eta,\delta)$, $P_1(\eta,\delta)$ and $P_2(\eta,\delta)$.
  If $B_k = (b^i{}_{,j})$ evaluated at P_k, then when the three steady states are distinct, B_0 and B_2 have real negative eigenvalues. B_1 has one real negative and one real positive eigenvalue. The eigenvector corresponding to the negative eigenvalue has positive slope.

- As η, δ vary, two of the steady states may coalesce and annihilate each other. This behavior is analogous to the marginal bifurcation.

• As η, δ vary, all three steady states may move together and coalesce when $\eta = \delta = 0$. At the critical bifurcation, $B_1 = (b^i_{,j})$ has a zero eigenvalue. We assume that the steady state remaining after the critical bifurcation is a stable steady state.

A deterministic system satisfying the above postulates will be structurally similar to the one sketched in Fig. 3.

The above properties can be restated in terms of a new coordinate system as follows. The y^1 axis is in the direction of the eigenvector of the non negative eigenvalue of B_1. The y^2 axis is in the direction of the eigenvector of the negative eigenvalue, with the origin at P_1. The deterministic evolution is then

$$\dot{y} = \tilde{b}(y, \eta, \delta). \tag{A.5}$$

A dynamical system is a critical type system if:

1) $\det(\tilde{b}^i{}_{,j}(P_1, 0, 0)) = 0$

2) $$\begin{aligned} \tilde{b}^1{}_{,1}(P_1,0,0) &= \tilde{b}^2{}_{,1}(P_1,0,0) \\ &= \tilde{b}^1{}_{,11}(P_1,0,0) = \tilde{b}^2{}_{,11}(P_1,0,0) = 0 \end{aligned} \tag{A.6}$$

3) $\tilde{b}^2{}_{,2}(P_1,0,0) \neq 0$

4) $\tilde{b}^1{}_{,111} - \tilde{b}^2{}_{,111} \neq 0$.

These conditions have the following interpretation 1) indicates that the system has a zero eigenvalue, while condition 3) indicates that the second eigenvalue is non-zero. Condition 2) indicates that the linear and quadratic dynamics in the y^1 direction vanish, while 4) indicates that the dynamics are cubic.

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